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The family of residue fields of a zero-dimensional commutative ring

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Abstract

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Given a zero-dimensional commutative ring R, we investigate the structure of the family $\mathscr{F}(R)$ of residue fields of R. We show that if a family $\mathscr{F}$ of fields contains a finite subset $\{F_1, \ldots, F_n\}$ such that every field in $\mathscr{F}$ contains an isomorphic copy of at least one of the F_i, then there exists a zero-dimensional reduced ring R such that $\mathscr{F} = \mathscr{F}(R)$. If every residue field of R is a finite field, or is a finite-dimensional vector space over a fixed field K, we prove, conversely, that the family $\mathscr{F}(R)$ has, to within isomorphism, finitely many minimal elements.

1. Introduction

All rings considered in this paper are assumed to be commutative and to contain a unity element. If R is a subring of a ring S, we assume that the unity of S is contained in R, and hence is the unity of R. If R is a ring and if $\{M_\alpha\}_{\alpha \in A}$ is the family of maximal ideals of R, we denote by $\mathscr{F}(R)$ the family $\{R/M_\alpha : \alpha \in A\}$ of residue fields of R and by $\mathscr{F}^*(R)$ a set of isomorphism-class representatives of $\mathscr{F}(R)$. In connection with work on the class of hereditarily zero-dimensional rings in [8], we encountered the problem of determining what families of fields can be realized in the form $\mathscr{F}(R)$ for some zero-dimensional ring R. This paper is

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devoted to an investigation of this question. Because of the breadth of the question, it was necessary that we sometimes place restrictions on the family of fields considered. Variants of the main problem of this paper have been considered by Popescu and Vraciu [12] and by Pierce [11].

If a family $\mathcal{F}$ of fields is of the form $\mathcal{F}(R)$ for some zero-dimensional ring R, then we say the family $\mathcal{F}$ is *realizable*. In general, if $\mathcal{F}$ and $\mathcal{G}$ are indexed families of fields, then by writing $\mathcal{F} = \mathcal{G}$, we mean that there is a bijection between the indexing sets of $\mathcal{F}$ and $\mathcal{G}$ that is such that corresponding elements under the bijection are isomorphic as fields. We use the symbol $\dot{\cup}$ to denote disjoint union.

Let $\mathcal{F} = \{K_\alpha\}_{\alpha \in A}$ be a family of fields. In Section 2 we prove some general results concerning realizability of $\mathcal{F}$. We observe in 2.5 that if R is any ring (not necessarily zero-dimensional), if $\mathrm{Spec}(R) = \{P_\lambda\}_{\lambda \in \Lambda}$, and if $K_\lambda = \mathcal{Q}(R/P_\lambda)$, then the family $\{K_\lambda\}_{\lambda \in \Lambda}$ is realizable. Theorem 2.11 shows that if there exists a finite subset $\{\alpha_1, \ldots, \alpha_m\}$ of A such that each K_α contains an isomorphic copy of at least one K_{α_i}, then $\mathcal{F}$ is realizable. Proposition 2.12 is an elementary, but basic, result that shows that if $\mathcal{F} = \dot{\bigcup}_{i \in I} \mathcal{F}_i$ is a partition by characteristic of $\mathcal{F}$ into nonempty subfamilies $\mathcal{F}_i$, then each $\mathcal{F}_i$ is realizable if $\mathcal{F}$ is realizable; the converse holds if I is finite, but not in general.

In Theorem 3.1 we prove that the converse of Theorem 2.11 holds in the case where each K_α is a finite-dimensional vector space over a fixed field K. We deduce (Corollary 3.2) that a family $\mathcal{F}$ of finite fields is realizable if and only if, to within isomorphism, $\mathcal{F}$ has only finitely many minimal elements.

Without the hypothesis that each of the fields K_α is finite-dimensional over K, Theorem 3.1 fails. In fact, Example 3.6 exhibits a realizable family $\mathcal{T} = \{K_i\}_{i=0}^{\infty}$ of algebraic extensions of $\mathrm{GF}(p)$ such that K_0 is the only infinite member of $\mathcal{T}$, while each K_i is minimal in $\mathcal{T}$. If $\mathcal{F}^*$ is a set of isomorphism-class representatives of the elements of $\mathcal{F}$, one question that arises in Section 3 (see 3.8) asks whether the families $\mathcal{F}$ and $\mathcal{F}^*$ are simultaneously realizable. If $\mathcal{F}$ satisfies the hypothesis of Theorem 3.1, then that result shows that $\mathcal{F}$ and $\mathcal{F}^*$ are simultaneously realizable; we know no example in which one of the families $\mathcal{F}$ or $\mathcal{F}^*$ is realizable, but the other is not.

Section 4 is concerned primarily with the problem of determining conditions under which the reverse implication in part (1) of Proposition 2.12 holds—that is, when $\mathcal{F}$ is realizable if each family $\mathcal{F}_i$ in its partition according to characteristic is realizable. The main result obtained in this direction is Theorem 4.14, which implies that if $\mathbb{Q}$ or $\mathbb{Q}(X_1, \ldots, X_n)$ (for some n) is in $\mathcal{F}$, then $\mathcal{F}$ is realizable if each $\mathcal{F}_i$ is realizable. The proof of Theorem 4.14 uses a gluing process for maximal ideals introduced by Doering and Lequain in [3].

Section 5 treats the case of the realizability question that was the basis of our initial interest in the topic—that is, the case of a family $\{K_\alpha\}$ with each K_α absolutely algebraic of nonzero characteristic.

In Section 6 we raise and briefly discuss two questions concerning uniqueness of realizability of a family of fields. If $\mathcal{F} = \mathcal{F}(R)$ is a realizable family, then the bijection between $\mathcal{F}$ and $\mathrm{Spec}(R)$ determines a topology on $\mathcal{F}$. We observe that for certain realizable families the topology defined in this way is unique, while for others it is not. We present in Example 6.4 an example due to Roger Wiegand that exhibits nonisomorphic zero-dimensional reduced rings R and S such that $\mathcal{F}(R) = \mathcal{F} = \mathcal{F}(S)$ by means of an identification that determines the same topology on $\mathcal{F}$.

Suppose R is a zero-dimensional reduced ring with prime spectrum $\{M_\alpha\}_{\alpha\in A}$. Because we frequently consider R as a subring of $\prod_\alpha(R/M_\alpha)$ via the diagonal imbedding, we review some of the terminology and basic results concerning the product $T=\prod_\alpha(K_\alpha)$ of a family $\{K_\alpha\}_{\alpha\in A}$ of fields. We denote by e_α the primitive idempotent with α-coordinate 1 and with each other coordinate 0. The direct sum ideal I of T is the ideal generated by $\{e_\alpha\}_{\alpha\in A}$; it consists of all elements of T with only finitely many nonzero coordinates. If M is a maximal ideal of T, then either (1) $I\subseteq M$, or (2) $e_\alpha \not\subseteq M$ for some α and $M=(1-e_\alpha)T$; in case (1) we say that M is a *free* maximal ideal of T, whereas M is said to be a *fixed* maximal of T if (2) is satisfied. This terminology comes from the theory of rings of continuous functions, where the elements of T are considered as functions from A into $\bigcup_{\alpha\in A}(K_\alpha)$ and an ideal is fixed if its elements have a common zero [5, p. 54]. We remark that there exist free maximal ideals of T if and only if the set A is infinite, cf. [5, (4.10)]. Moreover, each nonzero element of a free maximal ideal of T has infinitely many nonzero coordinates. It is well known that $T=\prod_\alpha(K_\alpha)$ is zero-dimensional and the association of K_α with $T/(1-e_\alpha)$ defines an injection of the family $\{K_\alpha\}_{\alpha\in A}$ into $\mathcal{F}(T)$. The existence of free maximal ideals, however, implies that this inclusion is strict if A is infinite.

2. General results on realizability of a family of fields

We begin by recording several elementary and basic facts concerning the realizability question. Recall that a family $\mathcal{F}$ of fields is realizable if $\mathcal{F} = \mathcal{F}(R)$ for some zero-dimensional ring R.

2.1. If R is a ring with nilradical N, then $\mathcal{F}(R)=\mathcal{F}(R/N)$. Therefore the realizability question for zero-dimensional reduced rings is equivalent to that for general zero-dimensional (commutative) rings.

2.2. If $F=\{F_1,\ldots,F_n\}$ is a finite family of fields and if $R=F_1\oplus\cdots\oplus F_n$, then $\mathcal{F}=\mathcal{F}(R)$. Therefore any finite family of fields is realizable. More generally, if $\mathcal{F}_1,\ldots,\mathcal{F}_n$ are families of fields that are realizable and if $\mathcal{F}$ is the disjoint union $\dot\bigcup_{i=1}^n \mathcal{F}_i$ of the families $\mathcal{F}_i$, then $\mathcal{F}$ is also realizable. In fact, if R_i is a zero-dimensional ring such that $\mathcal{F}_i=\mathcal{F}(R_i)$ and if $R=R_1\oplus\cdots\oplus R_n$, then R is zero-dimensional and $\mathcal{F}=\mathcal{F}(R)$.

2.3. In part (ii) of Remark 4.4 of [8] it is noted that if R is a zero-dimensional ring such that the set

$$\mathscr{C}(R) = \{\mathrm{char}(R/M)\colon M \text{ is a prime ideal of } R\}$$

is an infinite set, then $0 \in \mathscr{C}(R)$. To see this statement, observe that $\mathrm{char}(R) = 0$ since $\mathscr{C}(R)$ is infinite. Hence $\mathbb{Z} \subseteq R$, and because the zero ideal of $\mathbb{Z}$ is contracted from R, it is contracted from a prime ideal M of R, whence $\mathrm{char}(R/M) = 0$.

2.4. In [9], it is shown that if R is a subring of a zero-dimensional subring of a ring T, then there exists a unique minimal zero-dimensional subring of T containing R. If this minimal zero-dimensional extension of R in T is denoted R^0 and if for a prime ideal P of R we denote by $\mathscr{Q}(R/P)$ the quotient field of R/P, then it is shown in Theorem 3.3 of [9] that

$$\mathscr{F}(R^0) = \{\mathscr{Q}(R/P)\colon P \in \mathrm{Spec}(R) \text{ is contracted from } T\}\,.$$

2.5. In general, if R is any ring (not necessarily zero-dimensional), if $\mathrm{Spec}(R) = \{P_\lambda\}_{\lambda\in\Lambda}$, and if $K_\lambda = \mathscr{Q}(R/P_\lambda)$, then $\{K_\lambda\}_{\lambda\in\Lambda}$ is realizable. One way to verify this assertion is to first note that, as in 2.1, there is no loss of generality in assuming that R is reduced. Thus R can be considered as a subring of the product $T = \prod_{\lambda\in\Lambda}(K_\lambda)$ of fields under the diagonal embedding. Then P_λ is the contraction to R of the maximal ideal $(1 - e_\lambda)T$ of T consisting of tuples where the λ-coordinate is zero. By 2.4, $\mathscr{F}(R^0) = \{K_\lambda\}_{\lambda\in\Lambda}$. Alternatively, to show for any commutative ring R that the indexed family $\{K_\lambda\} = \{\mathscr{Q}(R/P_\lambda)\colon P_\lambda \in \mathrm{Spec}(R)\}$ is realizable, one can use the universal regular ring $\hat{R}$ associated to R [14, 15]. There is a well-defined ring homomorphism $\phi : R \to \hat{R}$, and it is shown in [15, Theorem 1] that: (i) $\hat{R}$ is a zero-dimensional reduced ring, (ii) the map $P \to \phi^{-1}(P)$ is a homeomorphism between $\mathrm{Spec}(\hat{R})$ and $\mathrm{Spec}(R)$ topologized with the patch topology[1], and ϕ induces an isomorphism between the quotient field of $R/\phi^{-1}(P)$ and the field $\hat{R}/P$. Roger Wiegand has pointed out to us that this gives the formally stronger result that for any ring R and any patch in $\mathrm{Spec}(R)$, the corresponding family of fields is realizable.

2.6. Taking $R = \mathbb{Z}$ in 2.5, it follows that the family of prime fields $\mathscr{F} = \{\mathrm{GF}(p)\colon p$ is a prime$\} \cup \{\mathbb{Q}\}$ is realizable (cf. [4, 11]). More generally, suppose $\mathscr{F}'$ is any subset of $\mathscr{F}$ that contains $\mathbb{Q}$. Let R' denote the localization of $\mathbb{Z}$ with respect to the multiplicative system generated by the prime integers p such that $\mathrm{GF}(p) \in \mathscr{F} - \mathscr{F}'$. Then $\mathscr{F}' = \{\mathscr{Q}(R'/P)\colon P \in \mathrm{Spec}(R')\}$, so by 2.5, the family $\mathscr{F}'$ is realizable.

Using 2.4 instead of 2.5, it follows that a family such as $\{K_i\} \cup \{\mathbb{Q}\}$, where $\{K_i\}$ contains exactly two copies of each finite prime field, is realizable. To see this, let $T = \prod_i(K_i)$, let π be the prime subring of T, I the direct sum ideal of T,

[1] To define the *patch* topology on the set $\mathrm{Spec}(R)$, one takes as an open subbase all compact open sets of $\mathrm{Spec}(R)$ in the Zariski topology and their complements. A *patch* is then defined to be a subset of $\mathrm{Spec}(R)$ that is closed in the patch topology.

and let $R = \pi + I$. (Note that $\pi \cong \mathbb{Z}$ and $R/I \cong \mathbb{Z}$.) Then $\operatorname{Spec}(R)$ consists of the maximal ideals $(1-e_i)R$, where e_i is the primitive idempotent with support $\{i\}$, together with I and the maximal ideals $p\pi + I$ with p a prime integer. The ideals $(1-e_i)R$ and I are clearly contracted from the zero-dimensional ring T. However, the maximal ideals $p\pi + I$ are not contracted from T since $pT + I = T$ for each p. Hence if R^0 is the minimal zero-dimensional subring of T containing R, then 2.4 implies that $\mathscr{F}(R^0) = \{R/(1-e_i)R\} \cup \{\mathscr{Q}(R/I)\} = \{K_i\} \cup \{\mathbb{Q}\}$. (For a more general statement, see Corollary 4.4.)

Theorem 2.7. *Suppose $\mathscr{F} = \{K\} \cup \{K_\alpha\}$ is a family of fields such that, for each $\alpha \in A$, there exists a monomorphism ϕ_α of K into K_α. Then $\mathscr{F}$ is realizable.*

Proof. We may assume that $\{K_\alpha\}$ is nonempty. We are not excluding the possibility that $K \cong K_\alpha$ for certain α, or that $K_\alpha \cong K_\beta$ for $\alpha \neq \beta$. Let $T = \prod_{\alpha \in A} K_\alpha$, let I be the direct sum ideal in T, let L be the imbedding of K in T via the monomorphism $\prod(\phi_\alpha)$, and let $R = L + I$. Since T is reduced, the subring R of T is reduced. For $\alpha \in A$, let e_α be the primitive idempotent associated with α. Then $1 - e_\alpha$ generates a maximal ideal M_α of R and $R/M_\alpha \cong K_\alpha$; also, I is a maximal ideal of R and $R/I \cong K$. We observe that $\operatorname{Spec}(R) = \{I\} \cup \{M_\alpha\}_{\alpha \in A}$. To prove this, take $P \in \operatorname{Spec}(R)$. If $I \subseteq P$, then $I = P$; if I is not contained in P, then $e_\alpha \not\in P$ for some α, and hence $1 - e_\alpha \in P$. Consequently, $M_\alpha \subseteq P$ and $P = M_\alpha$. Thus $\operatorname{Spec}(R) = \{I\} \cup \{M_\alpha\}_{\alpha \in A}$. Since there are no inclusion relations among these prime ideals, we conclude that R is zero-dimensional and $\mathscr{F}(R) = \{K\} \cup \{K_\alpha\}_{\alpha \in A}$. $\square$

Remark 2.8. The zero-dimensional reduced ring R constructed in the proof of Theorem 2.7 has at most one prime that is not finitely generated. We have

$$\mathscr{F}(R) = \{K\} \cup \{K_\alpha\}_{\alpha \in A} \quad \text{with } \operatorname{Spec}(R) = \{P\} \cup \{P_\alpha\}_{\alpha \in A},$$

where $R/P \cong K$ and $R/P_\alpha \cong K_\alpha$ for each α. The proof of Theorem 2.7 shows that $P_\alpha = (1 - e_\alpha)R$ for each α, while the ideal P is finitely generated if and only if A is finite. We remark that in general if R is a zero-dimensional reduced ring and $P \in \operatorname{Spec}(R)$ is a finitely generated prime ideal, then R/P is isomorphic to a direct summand of R. Therefore the family $\mathscr{F}(R) - \{R/P\}$ is realizable.

Example 2.9. There exist nonisomorphic zero-dimensional reduced rings R and S such that $\mathscr{F}(R) = \mathscr{F}(S)$. For example, if R is as in Theorem 2.7 with $\mathscr{F}(R)$ consisting of countably infinitely many copies of the Galois field GF(2) with two elements, and if $S = R \oplus R$, then each $\mathscr{F}(R)$ and $\mathscr{F}(S)$ consists of countably infinitely many copies of GF(2), but $R \not\cong S$ because S has two maximal ideals that are not finitely generated, while R has only one such maximal ideal.

Remark 2.10. The rings R and S of Example 2.9 are such that Spec(R) and Spec(S) are not homeomorphic as topological spaces with the Zariski topology because Spec(R) has a unique nonisolated point, while Spec(S) has two such points. In Section 6 we consider briefly two additional questions concerning uniqueness of realizability of a family $\mathcal{F}$ of fields.

Theorem 2.11. *Suppose $\mathcal{F} = \{K_\alpha\}_{\alpha \in A}$ is a family of fields. If there exists a finite subset $\{\alpha_1, \ldots, \alpha_m\}$ of A such that each $K_\alpha \in \mathcal{F}$ contains an isomorphic copy of at least one of the fields K_{α_i}, then the family $\mathcal{F}$ is realizable.*

Proof. We may assume that $\mathcal{F}$ is nonempty and that m is minimal with the stated property. Partition A into subsets $A_1, \ldots, A_m$, where $\alpha \in A_i$ if i is the smallest positive integer such that K_α contains an isomorphic copy of K_{α_i}. Minimality of m implies that each A_i is nonempty. By Theorem 2.7, there exists a zero-dimensional reduced ring R_i such that $\mathcal{F}(R_i) = \{K_\alpha : \alpha \in A_i\}$. By 2.2, it follows that if $R = R_1 \oplus \cdots \oplus R_m$, then $\mathcal{F}(R) = \mathcal{F}$. $\square$

Proposition 2.12. *Let $\mathcal{F}$ be a family of fields, and partition $\mathcal{F}$ into nonempty subsets $\mathcal{F}_i$, $i \in I$, where $\mathcal{F}_i$ consists of all elements of $\mathcal{F}$ of characteristic c_i (where c_i is 0 or a prime integer).*

(1) *If $\mathcal{F}$ is realizable, then each $\mathcal{F}_i$ is realizable.*

(2) *If each $\mathcal{F}_i$ is realizable and if I is finite, then $\mathcal{F}$ is realizable.*

(3) *If each $\mathcal{F}_i$ is realizable and I is infinite, then a necessary condition in order that $\mathcal{F}$ be realizable is that $\mathcal{F}$ contains a field of characteristic zero. Thus, even if each $\mathcal{F}_i$ is realizable, $\mathcal{F}$ need not be realizable.*

Proof. (1) Suppose $\mathcal{F} = \mathcal{F}(R)$, where R is zero-dimensional and reduced. If $c_i \neq 0$, then $\mathcal{F}_i = \mathcal{F}(R/c_iR)$ and if $c_i = 0$, $\mathcal{F}_i = \mathcal{F}(R_S)$, where S is the multiplicative system consisting of all nonzero integers; in either case, the rings R/c_iR and R_S are zero-dimensional.

(2) This follows from 2.2.

(3) The first statement in (3) follows from 2.3. If $\{p_i\}_{i=1}^{\infty}$ is the set of positive prime integers, then $\{\mathrm{GF}(p_i)\}$ is realizable for each i, but as we have just observed, $\{\mathrm{GF}(p_i)\}_{i=1}^{\infty}$ is not realizable. $\square$

In Section 4 we consider the problem of giving conditions under which $\mathcal{F}$ is realizable if each $\mathcal{F}_i$ is realizable. A principal result in this direction is Theorem 4.14, which shows that if each $\mathcal{F}_i$ is realizable, if $c_0 = 0$, and if $\mathbb{Q}$ is in $\mathcal{F}_0$, then $\mathcal{F}$ is realizable.

3. Families of finite algebraic extensions of a given field

Suppose $\mathcal{F} = \{K_\alpha\}_{\alpha \in A}$ is a family of fields. Theorem 2.11 gives a sufficient condition for $\mathcal{F}$ to be realizable. This sufficient condition is not necessary in general, but in Theorem 3.1 we show that if each K_α is a finite-dimensional algebra over a fixed field K, then the condition of Theorem 3.1 is also necessary for realizability of $\mathcal{F}$. A counterexample to general necessity of the condition of Theorem 3.1 is given in Example 3.6.

Theorem 3.1. *Suppose K is a field and R is a K-algebra such that R/M is a finite-dimensional K-algebra for each $M \in \operatorname{Spec}(R)$. There exists a finite subset $\{M_i\}_{i=1}^n$ of $\operatorname{Spec}(R)$ such that, for each $M \in \operatorname{Spec}(R)$, there exists i between 1 and n such that R/M_i is a K-subalgebra of R/M.*

Proof. By passage from R to R/N, where N is the nilradical of R, we assume without loss of generality that R is reduced. Since R/M is integral over K for each $M \in \operatorname{Spec}(R)$, it follows that R is integral over K [6, p. 227]. For $M \in \operatorname{Spec}(R)$, choose a set $Y(M) = \{\theta_i\}_{i=1}^m$ of representatives of the residue classes of M in R so that $\{\theta_i + M\}_{i=1}^m$ is a basis for R/M as a vector space over K. Then R/M and $K[Y(M)]/(M \cap K[Y(M)])$ are isomorphic as K-algebras and $M \cap K[Y(M)]$ is principal, generated by an idempotent e_M. Consider $\mathcal{D}(1-e_M) = \{P \in \operatorname{Spec}(R)\colon e_M \in P\}$; $\mathcal{D}(1-e_M)$ is an open subset of $\operatorname{Spec}(R)$ containing M. Moreover, $K[Y(M)] \cap P = K[Y(M)] \cap M = e_M K[(Y(M)]$ for each $P \in \mathcal{D}(1-e_M)$. Since $\operatorname{Spec}(R)$ is compact, we can choose $M_1, \ldots, M_n$ in $\operatorname{Spec}(R)$ such that $\{\mathcal{D}(1-e_{M_i})\}_{i=1}^n$ is an open cover of $\operatorname{Spec}(R)$ [1, Exercise 17, p. 12]. If $M \in \operatorname{Spec}(R)$, then for some i between 1 and n, $M \in \mathcal{D}(1-e_{M_i})$. Then R/M contains

$$K[Y(M_i)]/(K[Y(M_i)] \cap M) = K[Y(M_i)]/(K[Y(M_i)] \cap M_i) \simeq R/M_i$$

as a K-algebra. This completes the proof. □

If, in the notation of the proof of Theorem 3.1, we let $R_0 = K[Y(M_1), \ldots, Y(M_n)]$ be the compositum of the rings $R[Y(M_1)], \ldots, R[Y(M_n)]$, then R_0 is a finite-dimensional algebra over K, and we have the following alternate formulation of Theorem 3.1.

Theorem 3.1A. *If K is a field and R is a reduced K-algebra such that R/M is a finite-dimensional K-algebra for each $M \in \operatorname{Spec}(R)$, then there exists a finitely generated K-subalgebra R_0 of R and a finite subset $\{M_i\}_{i=1}^n$ of $\operatorname{Spec}(R)$ such that, for each $M \in \operatorname{Spec}(R)$, there exists i between 1 and n such that $R_0/(M \cap R_0)$, $R_0/(M_i \cap R_0)$ and R/M_i are isomorphic as K-subalgebras of R/M.*

Theorem 3.1 provides a partial converse of Theorem 2.11. It also implies the following corollary.

Corollary 3.2. *Let $\{K_\alpha\}_{\alpha \in A}$ be a family of (not necessarily distinct) finite fields. Then $\{K_\alpha\}_{\alpha \in A}$ is realizable if and only if there exists a finite subset $\{a_i\}_{i=1}^n$ of A such that each K_α contains, up to isomorphism, at least one of the fields K_{α_i}.*

Proof. Theorem 2.11 shows that the stated condition is sufficient. Conversely, if $\{K_\alpha\}$ is realizable, then since each K_α is finite, part (3) of Proposition 2.12 implies that a partition of $\mathcal{F}$ according to characteristic yields only finitely many equivalence classes $\mathcal{F}_i$. Moreover, each $\mathcal{F}_i$ is realizable. To show the stated condition is necessary, it suffices to show that each $\mathcal{F}_i$ satisfies the stated condition. Thus, let R be a zero-dimensional reduced ring such that $\mathcal{F}(R) = \mathcal{F}_i$. The prime subring K of R is a field isomorphic to the common subfield of the elements of $\mathcal{F}_i$, and hence Theorem 3.1 implies that $\mathcal{F}_i$ has, up to isomorphism, only finitely many minimal elements. □

In Example 3.6 we show that the conclusion of Theorem 3.1 may fail if at least one of the residue fields R/M is not finite-dimensional over K. (In Example 3.6, R/M is not finite-dimensional over K for a unique maximal ideal M of R, and this R/M is algebraic over K.)

The statement of the next result, Theorem 3.3, uses the following notation. If $\mathcal{U} = \{F_j \colon j \in J\}$ is a family of (not necessarily distinct) subfields of a field L, $C(\mathcal{U})$ is defined as the set of all elements $x \in L$ such that the set $\{j \in J \colon x \not\in F_j\}$ is finite. It is easy to check that $C(\mathcal{U})$ is a subfield of L; if $\mathcal{U} = \{F_i\}_{i=1}^{\infty}$, where $F_1 < F_2 < \cdots$, then $C(\mathcal{U}) = \bigcup_{i=1}^{\infty} F_i$.

Theorem 3.3. *Suppose K_j is a subfield of the field L for each j in the index set J, and let F_j be a subfield of K_j for each j in J. Let I be the direct sum ideal of $T = \prod_{j\in J} K_j$ and let R be the subring of T consisting of all tuples $\{a_j\}_{j\in J}$ in $\prod_{j\in J} F_j$ such that, for some $j_0 \in J$, $\{j \in J \colon a_j \neq a_{j_0}\}$ is finite (that is, all but finitely many of the coordinates of $\{a_j\}$ have the same value). If $S = R + I$, then S is zero-dimensional, reduced, and $\mathcal{F}(S) = \{K_j\} \cup \{C(\mathcal{U})\}$, where $\mathcal{U} = \{F_j\}$.*

Proof. It is clear that S is reduced. We determine Spec(S). For each $j \in J$, the primitive idempotent e_j is in $I \subseteq S$, so $(1 - e_j)S$ is maximal in S with residue field K_j. Suppose $P \in \mathrm{Spec}(S)$ is distinct from each $(1 - e_j)S$. If $y \in I$, then y is annihilated by $(1 - e_{j_1}) \cdots (1 - e_{j_n})$ for some $j_1, \ldots, j_n \in J$, and hence $y \in P$ and $I \subseteq P$. To complete the proof, it therefore suffices to show that I is maximal in S and $S/I \simeq C(\mathcal{U})$. Thus, if $s = r + x \in S$, where $r \in R$ and $x \in I$, then all but finitely many of the coordinates of s have the same value v_s; v_s is the value of all but finitely many of the coordinates of r (that is, $v_s = v_r$), and hence $v_s \in C(\mathcal{U})$. It is then straightforward to show that the map $s \to v_s$ is a homomorphism from S onto $C(\mathcal{U})$ with kernel I, and this completes the proof. □

Theorem 3.3 provides a new method for proving realizability of a family $\mathcal{G} = \{K_\alpha\}_{\alpha\in A}$ of subfields of a field L. For short we will say that $\mathcal{G}$ is *$*$-realizable* if there exists $\alpha \in A$ and a subfield F_β of K_β for each $\beta \in A$, $\beta \neq \alpha$, such that $K_\alpha = C(\{F_\beta\})$. Theorem 3.4 gives equivalent conditions for $\mathcal{G}$ to be $*$-realizable.

Theorem 3.4. *Suppose $\mathscr{G} = \{K_\alpha\}_{\alpha \in A}$ is a family of subfields of the field L, where A is infinite. Then $\mathscr{G}$ is $*$-realizable if and only if $C(\mathscr{G})$ contains an element of $\mathscr{G}$.*

Proof. Suppose first that $C(\mathscr{G})$ contains K_α, where $\alpha \in A$. For $\beta \in A$, $\beta \neq \alpha$, let $F_\beta = K_\beta \cap K_\alpha$. If $\mathscr{F} = \{F_\beta\}$ then $C(\mathscr{F}) \subseteq K_\alpha$ since each F_β is contained in K_α. Moreover, if $x \in K_\alpha \subseteq C(\mathscr{G})$, then $\{\beta: x \not\in F_\beta\} = \{\beta: x \not\in K_\beta\}$ is a finite set, so $x \in C(\mathscr{F})$ and equality holds: $K_\alpha = C(\mathscr{F})$. Consequently, $\mathscr{G}$ is $*$-realizable.

Conversely, if $\alpha \in A$ is such that $K_\alpha = C(\{F_\beta\})$, where F_β is a subfield of K_β for each $\beta \in A \backslash \{\alpha\}$, then $K_\alpha \subseteq C(\mathscr{G})$ since it is clear that $C(\{F_\beta\}) \subseteq C(\mathscr{G})$. □

Corollary 3.5. *If $\mathscr{G} = \{K_\alpha\}_{\alpha \in A}$ is a family of subfields of the field L and if $C(\mathscr{G}) \in \mathscr{G}$, then $\mathscr{G}$ is $*$-realizable.* □

A family may be realizable, but not $*$-realizable. For example, the family

$$\mathscr{G} = \{\mathrm{GF}(p^{2^i})\}_{i=1}^{\infty} \cup \{\mathrm{GF}(p^{3^i})\}_{i=1}^{\infty}$$

is realizable by Theorem 2.11, but $C(\mathscr{G}) = \mathrm{GF}(p)$, so $\mathscr{G}$ is not $*$-realizable by Theorem 3.4.

Identifying via isomorphism, we see that Theorem 3.4 can be extended to any family $\mathscr{F} = \{F_\alpha\}$ such that each F_α is imbeddable in a fixed field L. If $\mathscr{F}$ has this property, then each F_α in $\mathscr{F}$ has the same characteristic. The converse also holds: if each F_α has the same prime subfield π, then each F_α is imbeddable in the algebraic closure of a pure transcendental extension of π of appropriately large transcendence degree over π. Hence Theorem 3.4 extends to a family of fields, all of the same characteristic.

Example 3.6. There exists a zero-dimensional ring R such that $\mathscr{F}(R) = \{K_i\}_{i=0}^{\infty}$, where the K_i are pairwise incomparable fields all of the same characteristic, all algebraic over a finite field, and all but one of which are finite fields. We use the following notation: d is a positive integer, p is prime, L is an algebraic closure of $\mathrm{GF}(p)$, $\{q_i\}_{i=1}^{\infty}$ is a sequence of odd primes, $K_i = \mathrm{GF}(p^{d2^i q_i})$ for each $i \in \mathbb{Z}^+$, and F_i is the subfield $\mathrm{GF}(p^{d2^i})$ of K_i. Theorem 3.3 shows that the family $\mathscr{F} = \{K_i\}_{i=1}^{\infty} \cup \{(\bigcup_{i=1}^{\infty} F_i) = K_0\}$ is realizable. Each field in this family is algebraic over $\mathrm{GF}(p)$, each K_i except K_0 is finite-dimensional over $\mathrm{GF}(p)$, and no field in $\mathscr{F}$ is imbeddable in a member of $\mathscr{F}$ distinct from itself.

3.7. We remark that an example similar to Example 3.6 can be obtained in characteristic 0 by taking L to be an appropriate subfield of an abelian closure A of $\mathbb{Q}$. If q is any prime, it is known ([13], Sections 7.3 and 13.1] and [7, Example 2.7]) that there exists an ascending sequence $K_1 < K_2 < \cdots$ of subfields of A so that K_i is cyclic over $\mathbb{Q}$ of degree q^i for each i. If L_q is the union of this chain of fields K_i, then the only proper extensions of $\mathbb{Q}$ in L_q are the fields K_i and the field

L_q itself [13, Proposition 13.1]. If L is the compositum of the family $\{L_q: q$ is prime$\}$, then the structure of the set of intermediate fields between $\mathbb{Q}$ and L is entirely analogous to that of the fields between the field $GF(p)$ and its algebraic closure; here the fields L_q correspond to the fields $GF(p^{q^\infty})$ in characteristic p. One difference in the analogy is that the field L is real, and hence is a proper subfield of A (see, for example, [7, p. 87]).

3.8. Suppose $\mathcal{F}$ is a family of fields and $\mathcal{F}^*$ is a family of representatives of the isomorphism classes of $\mathcal{F}$. It is natural to ask about the relation between realizability of $\mathcal{F}$ and realizability of $\mathcal{F}^*$. We know no example where one of these families is realizable and the other is not. Theorem 3.1 implies that if there exists a field K such that each member of $\mathcal{F}$ is a finite-dimensional K-algebra, then $\mathcal{F}$ and $\mathcal{F}^*$ are simultaneously realizable. In particular, Corollary 3.2 shows that if each member of $\mathcal{F}$ is finite-dimensional over a fixed prime field, then $\mathcal{F}$ and $\mathcal{F}^*$ are simultaneously realizable.

4. Realizability of a family partitioned according to characteristic

Let $\mathcal{T}$ be a family of fields and let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \ldots$ be a partition of $\mathcal{T}$ according to characteristic, as in Proposition 2.12. In considering realizability of $\mathcal{T}$, part (1) of Proposition 2.12 leads naturally to consideration of realizability of $\mathcal{T}_i$, and several results in Section 3 concern this case. On the other hand, part (3) of Proposition 2.12 shows that $\mathcal{T}$ need not be realizable if each $\mathcal{T}_i$ is realizable. Several results of this section (for example, Theorems 4.8, 4.14 and 4.15) give sufficient conditions for $\mathcal{T}$ to be realizable if each $\mathcal{T}_i$ is realizable. In particular, if $\mathcal{T}_0$ is the collection of elements of $\mathcal{T}$ of characteristic 0, then Theorem 4.14 shows that $\mathcal{T}$ is realizable if each $\mathcal{T}_i$ is realizable and $\mathbb{Q}$ or $\mathbb{Q}(X_1, \ldots, X_n)$, for some n, is in $\mathcal{T}_0$.

We begin with a preliminary result that will be used throughout this section.

Lemma 4.1. *Suppose R is a subring of the ring T, e is an idempotent of T, and $Te \subseteq R$. If P is a prime ideal of R containing $1-e$, then $P^* = Pe \oplus T(1-e)$ is prime in T and is the unique prime of T lying over P in R. Moreover, $R/P \simeq T/P^* \simeq Te/Pe$.*

Proof. We have $R = Re \oplus R(1-e)$ and $T = Te \oplus T(1-e)$. Since $1-e \in P$, $P = (P \cap Re) \oplus R(1-e) = Pe \oplus R(1-e)$, where Pe is prime in Re. Since $Te \subseteq R$, we have $Te \cdot e \subseteq Re$—that is, $Te \subseteq Re$ and hence $Te = Re$. Consequently, $P^* = Pe \oplus T(1-e)$ is prime in $Te \oplus T(1-e) = T$. Clearly $P^* \cap R = Pe + R(1-e) = P$. If Q is any prime of T lying over P in R, then $1-e \in Q$ so $Q = Qe \oplus T(1-e)$, where Qe is prime in Te and $Qe \cap Re = Pe$. On the other hand, since $Re = Te$, $Qe \cap Re = Qe \cap Te = Qe$. Therefore $Pe = Qe$ and $Q = Pe + T(1-e) = P^*$. It is clear that $T/P^* \simeq Te/Pe = Re/Pe \simeq R/P$. □

Proposition 4.2. *Let $\{T_\alpha\}_{\alpha\in A}$ be a family of zero-dimensional reduced rings, let $T=\prod_\alpha T_\alpha$, let I be the direct sum ideal of T, and let S be a zero-dimensional subring of T/I. If R is the inverse image of S under the canonical homomorphism from T onto T/I, then R is zero-dimensional and reduced and $\mathcal{F}(R)=[\dot{\bigcup}\,\mathcal{F}(T_\alpha)]\,\dot{\cup}\,\mathcal{F}(S)$.*

Proof. Since T is reduced, R is reduced. To see that R is zero-dimensional, choose $P\in\operatorname{Spec}(R)$. If $I\subseteq P$, then P/I is prime, hence maximal, in S so P is maximal in R. If $I\not\subseteq P$, then $1-e_\alpha\in P$ for some α. Since $Te_\alpha\subseteq R$, Lemma 4.1 shows that R/P is isomorphic to a residue class ring of $Re_\alpha\simeq R_\alpha$. Hence R/P is a field, P is maximal in R, and R is zero-dimensional.

Since $I\subseteq R$, it follows from Lemma 4.1 that $\dot{\bigcup}\,\mathcal{F}(T_\alpha)$ is the family of residue fields of maximal ideals of R that do not contain I. Moreover, since $R/I\simeq S$, $\mathcal{F}(S)$ is the family of residue fields of maximal ideals of R that contain I. This completes the proof. □

Corollary 4.3. *Suppose $\{K_\alpha\}$ is a family of fields, $T=\prod_\alpha K_\alpha$, I is the direct sum ideal of T, and the field L is a subfield of T/I. Then $\{K_\alpha\}\,\dot{\cup}\,\{L\}$ is realizable.* □

Corollary 4.4. *Suppose $\mathcal{T}=\{K_\alpha\}_{\alpha\in A}$ is a family of fields, where A is infinite and, for each prime p, the set $\{\alpha\in A\colon \operatorname{char}(K_\alpha)=p\}$ is finite. Then the family $\mathcal{T}\,\dot{\cup}\,\{\mathbb{Q}\}$ is realizable.*

Proof. Let I be the direct sum ideal of $T=\prod_\alpha K_\alpha$. In view of Corollary 4.3, it suffices to show that $\mathbb{Q}$ is imbedded in T/I. The hypotheses on A imply that $\mathbb{Z}$ is the prime subring of T, and if n is a nonzero element of $\mathbb{Z}$, only finite many coordinates of n are zero in T. Hence $n+I$ is a unit of T/I and $\mathbb{Q}$ is imbedded in T/I as desired. □

We remark that Corollary 4.12 generalizes Corollary 4.4 to the case where $\mathcal{T}$ is a union of certain realizable subfamilies $\mathcal{T}_i$. This generalization is obtained by showing that under appropriate hypotheses, the rational function field $\mathbb{Q}(X_1,\ldots,X_n)$ in n variables over $\mathbb{Q}$ is imbeddable in T/I.

4.5. In connection with the proof of Theorem 4.8, we observe that if M is a maximal ideal of a zero-dimensional reduced ring R and if M contains an element m that belongs to only finitely many maximal ideals of R, then M is finitely generated, hence principal and generated by an idempotent. In particular, if only finitely many maximals $P_1,\ldots,P_n$ of R are such that their associated residue field has a fixed nonzero characteristic p, then each P_i is principal.

Proposition 4.6 is another basic result concerning zero-dimensional reduced rings.

Proposition 4.6. *Suppose R is a zero-dimensional reduced ring with only finitely many maximal ideals $M_1, M_2, \ldots, M_k$ that are not finitely generated. If $m \in \bigcap_{i=1}^{k} M_i$, then m belongs to all but finitely many of the maximal ideals of R.*

Proof. Let e be the idempotent generator of mR and consider the decomposition $R = Re \oplus R(1-e)$ of R. The maximal ideals of R that do not contain m are in one-to-one correspondence with the maximal ideals of the ring Re, and since each nonfinitely generated maximal of R contains m, each maximal ideal of Re is finitely generated. Because Re is zero-dimensional, it follows that Re has only finitely many maximal ideals. Therefore only finitely many maximals of R fail to contain m. □

Suppose $\mathscr{T} = \{K_\alpha\}_{\alpha \in A}$ is a family of fields of nonzero characteristic such that A is infinite and, for each prime p, the set $\{\alpha \in A: \mathrm{char}(K_\alpha) = p\}$ is finite. If $\{F_i\}_{i=1}^n$ is a finite family of finite-dimensional extensions of $\mathbb{Q}$, Theorem 4.8 gives equivalent conditions for the family $\mathscr{T} \cup \{F_i\}_{i=1}^n$ to be realizable. The next result, Proposition 4.7, is a weakened form of Theorem 4.8 that will be used in the proof of that result.

Proposition 4.7. *Let $\mathscr{T} = \{K_\alpha\}_{\alpha \in A}$ be as in the preceding paragraph and let $K = \mathbb{Q}(\theta)$ be a finite-dimensional extension of $\mathbb{Q}$, where θ satisfies the irreducible monic polynomial $f(X)$ over $\mathbb{Z}$. Let $B = \{\alpha \in A$: $f(X)$ has a root in $K_\alpha\}$. If the set $A - B$ is finite, then $\mathscr{T} \cup \{K\}$ is realizable.*

Proof. Let $T = \prod_\alpha K_\alpha$ and let I be the direct sum ideal of T. The proof of Corollary 4.4 shows that $\mathbb{Q}$ is imbedded in T/I. We let R be the inverse image of $\mathbb{Q}$ under the canonical map from T onto T/I; thus R is zero-dimensional reduced and $\mathscr{F}(R) = \mathscr{T} \cup \{\mathbb{Q}\}$ (Proposition 4.2). For each $\beta \in B$, let y_β be a root of $f(X)$ in K_β and let $y = \{y_\alpha\}_{\alpha \in A}$, where $y_\alpha = 0$ for each $\alpha \in A - B$. The element y is integral over R since $yf(y) = 0$, so $R[y] = S$ is zero-dimensional and reduced. We claim that $\mathscr{F}(S) = \mathscr{T} \cup \{K\}$. Because $I \subseteq S$, Lemma 4.1 shows that $\mathscr{T}$ is the family of residue fields of maximal ideals of S that fail to contain I. On the other hand, $S/I = R[y]/I \simeq (R/I)[y + I]$, where $R/I \simeq \mathbb{Q}$ and $y + I$ satisfies the irreducible polynomial $f(X) \in (R/I)[X]$. It follows that $S/I \simeq Q[X]/(f(X)) \simeq K$. Therefore I is maximal in S and $\mathscr{F}(S) = \mathscr{T} \cup \{K\}$. □

Theorem 4.8. *Suppose $\{F_i\}_{i=1}^n$ is a finite collection of finite-dimensional extensions of $\mathbb{Q}$, say $F_i = \mathbb{Q}(\theta_i)$, where θ_i satisfies an irreducible monic polynomial $f_i(X) \in \mathbb{Z}[X]$. Let $f = f_1 f_2 \cdots f_n$ and let $\mathscr{T} = \{K_\alpha\}_{\alpha \in A}$ be a family of fields, each of nonzero characteristic, such that A is infinite and, for each prime p, $\{\alpha \in A$: $\mathrm{char}(K_\alpha) = p\}$ is finite. Then $\mathscr{T} \cup \{F_i\}_{i=1}^n$ is realizable if and only if the set $B = \{\alpha \in A$: $f(X)$ has no root in $K_\alpha\}$ is finite.*

Proof. Suppose first that $\mathcal{T} \cup \{F_i\}_{i=1}^n$ is realizable, say $\mathcal{T} \cup \{F_i\}_1^n = \mathcal{F}(R)$, where R is zero-dimensional and reduced. Let $\operatorname{Spec}(R) = \{M_\alpha\}_{\alpha \in A} \cup \{P_i\}_1^n$, where $R/M_\alpha \simeq K_\alpha$ and $R/P_i \simeq F_i$. The hypothesis on A implies that each M_α is principal, and Proposition 4.6 shows that each element of $\bigcap_{i=1}^n P_i$ belongs to all but finitely many of the ideals M_α. Choose $r_i \in R$ so that $r_i + P_i$ satisfies $f_i(X)$—that is, $f_i(r_i) \in P_i$. If $r \in R$ is chosen so that $r \equiv r_i \pmod{P_i}$ for $1 \le i \le n$, then $f_i(r) \in P_i$ for each i and $f(r) = f_1(r) \cdots f_n(r) \in \bigcap_1^n P_i$. Therefore $f(r) \in M_\alpha$ for all but finitely many elements $\alpha \in A$, which means that $r + M_\alpha$ is a root of $f(X)$ in $R/M_\alpha \simeq K_\alpha$. We conclude that the set B is finite, as asserted.

Conversely, suppose the set B is finite. Then $\{K_\alpha : \alpha \in B\}$ is realizable by 2.2, and hence to show that $\{K_\alpha\}_{\alpha \in A} \cup \{F_i\}_1^n$ is realizable, it suffices to show that $\{K_\alpha : \alpha \in A - B\} \cup \{F_i\}_1^n$ is realizable. Without loss of generality we therefore assume that B is empty, and hence that for each $\alpha \in A$, at least one of the polynomials $f_1, f_2, \ldots, f_n$ has a root in K_α. We partition the set A into subsets $A_1, \ldots, A_n$ as follows: $A_1 = \{\alpha \in A : f_1 \text{ has a root in } K_\alpha\}$. If $1 < i \le n$ and if sets $A_1, A_2, \ldots, A_{i-1}$ have been defined, we let $A_i = \{\alpha \in A - (\bigcup_{j=1}^{i-1} A_j) : f_i \text{ has a root in } K_\alpha\}$. Let $\mathcal{U}_i = \{K_\alpha : \alpha \in A_i\} \cup \{F_i\}$. Then $\mathcal{T} \cup \{F_i\}_1^n$ is the disjoint union of the sets $\mathcal{U}_i$, and we claim that each $\mathcal{U}_i$ is realizable; if A_i is finite, realizability of $\mathcal{U}_i$ is clear, and if A_i is infinite, $\mathcal{U}_i$ is realizable by Proposition 4.7. Therefore $\mathcal{T} \cup \{F_i\}_1^n = \bigcup \mathcal{U}_i$ is realizable by 2.2. This completes the proof of Theorem 4.8. □

Let $\mathcal{T} = \{K_\alpha\}_{\alpha \in A}$ be as in the statement of Theorem 4.8 and let $\{L_i\}_{i=1}^n$ be any finite collection of fields of characteristic 0. If the family $\mathcal{T} \cup \{L_i\}_{i=1}^n$ is realizable, the proof of Theorem 4.8 shows that for any polynomial $g(X)$ over $\mathbb{Z}$ with a root in each L_i, the set $\{\alpha \in A : g(X) \text{ has no root in } K_\alpha\}$ is finite. This observation implies, for example, that if L is an algebraic extension of $\mathbb{Q}$ such that the family $\{\mathrm{GF}(p) : p \text{ is prime}\} \cup \{L\}$ is realizable, then $L = \mathbb{Q}$; this is true because each irreducible polynomial over $\mathbb{Z}$ of degree greater than 1 fails to have a root modulo p for infinitely many primes p [10, Corollary 16.6.2, p. 153][2].

In contrast with the situation concerning realizability of, for example,

$$\{\mathrm{GF}(p) : p \text{ is prime}\} \cup \{L\}$$

in the case where L is algebraic over $\mathbb{Q}$, we proceed to show in Corollary 4.12 that families such as $\mathcal{T} \cup \{Q(X_1, \ldots, X_n)\}$, with $\mathcal{T}$ as in the statement of Theorem 4.8, are realizable.

Lemma 4.9. *Let n be a positive integer. For a prime integer p and for any n-tuple $\sigma = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, denote by $M(p, \sigma)$ the maximal ideal $(p, X_1 - a_1, \ldots, X_n - a_n)$ of $\mathbb{Z}[X_1, \ldots, X_n]$.*

[2] We are grateful to Dennis Estes for informing us of this result from [10].

(1) *If* $f \in \mathbb{Z}[X_1, \ldots, X_n]$, $f \neq 0$, *then there exist only finitely many primes* p *such that* $f \in M(p, \sigma)$ *for each* $\sigma \in \mathbb{Z}^n$.

(2) *If* H *is an infinite set of prime integers, there exist* $q_1, q_2, \ldots \in H$ *and* $\sigma_1, \sigma_2, \ldots \in \mathbb{Z}$ *such that* $\bigcap_{i=1}^{\infty} M(q_i, \sigma_i) = (0)$.

Proof. (1) If $\sigma = (a_1, \ldots, a_n)$, then $f \in M(p, \sigma)$ if and only if $f(a_1, \ldots, \alpha_n)$ is divisible by p. Thus, choose $\mu = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ such that $f(b_1, \ldots, b_n) \neq 0$. Only finitely many primes $p_1, \ldots, p_s$ of $\mathbb{Z}$ divide $f(b_1, \ldots, b_n)$, and hence $f \not\in M(p, \mu)$ for each prime p distinct from each p_i.

(2) Let $\{f_i\}_{i=1}^{\infty}$ be the sequence of nonzero elements of $\mathbb{Z}[X_1, \ldots, X_n]$. Part (1) shows that there exists $q_1 \in H$ and $\sigma_1 \in \mathbb{Z}^n$ such that $f_1 \not\in M(q_1, \sigma_1)$. Suppose n-tuples $\sigma_1, \ldots, \sigma_k$ and distinct primes $q_1, \ldots, q_k \in H$ have been chosen so that $f_i \not\in M(q_i, \sigma_i)$ for each i. It follows from (1) that there exists q_{k+1} in H and $\sigma_{k+1} \in \mathbb{Z}^n$ such that q_{k+1} is distinct from each q_i, $1 \leq i \leq k$, and $f_{k+1} \not\in M(q_{k+1}, \sigma_{k+1})$. By induction it follows that q_i and σ_i exist for all i, and since each nonzero element of $\mathbb{Z}[X_1, \ldots, X_n]$ is an f_i, $\bigcap_{i=1}^{\infty} M(q_i, \sigma_i) = (0)$. □

4.10. Suppose $\{I_\alpha : \alpha \in A\}$ is a family of nonzero ideals of an integral domain D. If $\bigcap I_\alpha = (0)$ and if d is a nonzero element of D, the set $\{\alpha \in A: d \not\in I_\alpha\}$ must be infinite, for if it were finite and consisted of elements $\alpha_1, \ldots, \alpha_n \in A$, then for any nonzero elements $x_i \in I_{\alpha_i}$, $dx_1x_2 \cdots x_n$ would be a nonzero element of $\bigcap I_\alpha$, contradicting the hypothesis. We use this observation in the proof of Theorem 4.11.

Theorem 4.11. *Let* $H = \{q_i\}_{i=1}^{\infty}$ *be an infinite set of distinct primes and for each* i, *let* T_i *be a zero-dimensional reduced ring of characteristic* q_i. *If* $T = \prod_{i=1}^{\infty} T_i$, *if* I *is the direct sum ideal of* T, *and if* n *is a positive integer, then* $\mathbb{Q}(X_1, \ldots, X_n)$ *is isomorphic to a subring of* T/I.

Proof. We denote by π the prime subring of T; since H is infinite, $\pi \simeq \mathbb{Z}$. Part (2) of Lemma 4.9 shows that there exists a subsequence $\{q_{i_j}\}_{j=1}^{\infty}$ of $\{q_i\}_{i=1}^{n}$ and elements $\sigma_{i_1}, \ldots, \sigma_{i_j}, \ldots$ in $\mathbb{Z}^n$ so that $\bigcap_{j=1}^{\infty} M(q_{i_j}, \sigma_{i_j}) = (0)$. We construct n elements $t_1, t_2, \ldots, t_n$ of T as follows: the i_j-entry of t_k is the k-coordinate of σ_{i_j} for $1 \leq k \leq n$, and if $i \in \mathbb{Z}^+ - \{i_j\}_{j=1}^{\infty}$, the i-coordinate of t_k is 0. Consider $g \in \pi[X_1, \ldots, X_n]$, $g \neq 0$. The i-coordinate of $g(t_1, \ldots, t_n)$ is $g(t_{1i}, \ldots, t_{ni})$, where t_{ji} is the i-coordinate of t_j. Since $g \neq 0$, it follows from Lemma 4.9 that there exist infinitely many integers j such that $g \not\in M(q_{i_j}, \sigma_{i_j})$, which means that q_{i_j} does not divide $g(\sigma_{i_j})$, the i_j-coordinate of $g(t_1, \ldots, t_n)$. Because the ring T_{i_j} has characteristic q_{i_j}, the i_j-coordinate of $g(t_1, \ldots, t_n)$ is nonzero, and hence $g(t_1, \ldots, t_n) \not\in I$. It follows that if $S = \pi[t_1, \ldots, t_n] + I$, then I is prime in S and $S/I \simeq \mathbb{Z}[X_1, \ldots, X_n]$. Let S^0 be the minimal zero-dimensional subring of T containing S. Since I is contracted from T, it follows from 2.4 that

$\mathbb{Q}(X_1, \ldots, X_n) \simeq \mathcal{Q}(S/I)$ is a residue field of S^0. Thus $\mathbb{Q}(X_1, \ldots, X_n)$ is isomorphic to a subring of T/I, as asserted. □

Corollary 4.12. *Let $H = \{q_i\}_{i=1}^{\infty}$ be an infinite family of distinct primes and for each i let $\mathcal{T}_i$ be a realizable family of fields of characteristic q_i. Then $(\bigcup \mathcal{T}_i) \cup \{\mathbb{Q}\}$ and $(\bigcup \mathcal{T}_i) \cup \{\mathbb{Q}(X_1, \ldots, X_n)\}$ are realizable for any $n \in \mathbb{Z}^+$.*

Proof. If T_i is a zero-dimensional reduced ring that realizes $\mathcal{T}_i$, then T_i has characteristic q_i. Hence if $T = \prod_i (T_i)$ and if I is the direct sum ideal of T, Theorem 4.11 shows that, to within isomorphism, $\mathbb{Q}$ and $\mathbb{Q}(X_1, \ldots, X_n)$ are subrings of T/I. The conclusions of Corollary 4.12 then follow from Proposition 4.2. □

Doering and Lequain in [3] introduced what they call "a gluing process for maximal ideals". In combination with Corollary 4.12, their process has relevance for the realizability question. We proceed to recall a special case of Theorem A of [3]. Suppose S is a zero-dimensional ring with maximal ideals M and P such that $S/M \simeq S/P \simeq K$. Let μ and ρ be homomorphisms from S onto K with kernels M and P, respectively, and let $\{M_\alpha\}_{\alpha \in A}$ be the set of maximal ideals of S that are distinct from M and P. If R is the subring of S consisting of all elements $s \in S$ such that $\mu(s) = \rho(s)$, then S is integral over R, M and P have the same contraction (namely $M \cap P$) to R, while M_α is the unique maximal ideal of S lying over $M_\alpha \cap R$ in R for each $\alpha \in A$. Moreover, $R/(M \cap P) \simeq K$ and $S/M_\alpha \simeq R/(M_\alpha \cap R)$ for each α. In terms of residue fields, the difference between $\mathcal{F}(S)$ and $\mathcal{F}(R)$ is that the two copies of K that arise in $\mathcal{F}(S)$ from M and P have been reduced to one in $\mathcal{F}(R)$; in [3], the maximal ideals M and P are said to be *glued over $M \cap P$ in R*. Of course, if K occurs n times as a residue field of S, then repetition of the process described above can be used to reduce the number of occurrences of K as a residue field by $n - 1$. Two applications of the gluing process are contained in Proposition 4.13 and Theorem 4.14.

Proposition 4.13. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are realizable families of fields and that there exist $F_1, \ldots, F_n \in \mathcal{F}$ and $K_1, \ldots, K_n \in \mathcal{G}$ such that $F_i \simeq K_i$ for each i. Then the family $\{\mathcal{F} - \{F_i\}_{i=1}^n\} \cup \mathcal{G}$ is realizable.*

Proof. 2.2 shows that $\mathcal{F} \cup \mathcal{G}$ is realizable, and by use of the gluing process we can reduce by one each of the occurrences F_i, K_i of F_i in $\mathcal{F} \cup \mathcal{G}$ in a zero-dimensional ring. □

Theorem 4.14. *Let H be a set of distinct primes, and for each $p \in H$, let $\mathcal{T}_p$ be a realizable family of fields of characteristic p. Let $\mathcal{T}_0$ be a realizable family of fields of characteristic 0, If $\mathbb{Q}$ or $\mathbb{Q}(X_1, \ldots, X_n)$, for some n, is in $\mathcal{T}_0$, then $\mathcal{T}_0 \cup \{\mathcal{T}_p \colon p \in H\}$ is realizable.*

Proof. We give the proof for $\mathbb{Q}$; the proof for $\mathbb{Q}(X_1, \ldots, X_n)$ is similar. Thus, Corollary 4.12 shows that $(\bigcup_{p\in H}\mathcal{T}_p)\cup\{\mathbb{Q}\}$ is realizable, and Proposition 4.13 then implies that $(\bigcup_{p\in H}\mathcal{T}_p)\cup\mathcal{T}_0$ is also realizable. □

Pierce in [11] has introduced the notion of a *minimal regular ring*. An equivalent form of the definition (see [11, Proposition 2.2]) states that a ring R is minimal regular if and only if R is zero-dimensional, reduced, and R/M is a prime field for each $M\in\mathrm{Spec}(R)$. Theorem 4.15 allows us to settle completely the realizability question for minimal regular rings—that is, for a family consisting of prime fields.

Theorem 4.15. *Let $\mathcal{T}=\{K_\alpha\}_{\alpha\in A}$ be a family of prime fields and let $\mathcal{T}^*$ be a set of isomorphism-class representatives of the elements of $\mathcal{T}$. Then $\mathcal{T}$ is realizable if and only if either $\mathcal{T}^*$ is finite, or else $\mathcal{T}^*$ is infinite and $\mathbb{Q}\in\mathcal{T}^*$.*

Proof. If $T^*=\{F_i\}_{i=1}^n$ is finite with n distinct elements, then partition $\mathcal{T}$ into isomorphism classes $\mathcal{T}_1,\ldots,\mathcal{T}_n$, where each member of $\mathcal{T}_i$ is isomorphic to F_i. Theorem 2.7 implies that each $\mathcal{T}_i$ is realizable, and hence $\mathcal{T}$ is realizable by 2.2. If $\mathcal{T}^*$ is infinite and contains $\mathbb{Q}$, we again partition $\mathcal{T}$ into isomorphism classes $\mathcal{T}_0,\mathcal{T}_1,\ldots,$ where $\mathcal{T}_0=\{K_\alpha\colon \alpha\in A$ and $K_\alpha\simeq\mathbb{Q}\}$. As before, each $\mathcal{T}_i$ is realizable, and Theorem 4.14 shows that $\mathcal{T}=\bigcup\mathcal{T}_i$ is also realizable.

To prove the converse we need only show that if $\mathcal{T}^*$ is infinite and $\mathbb{Q}\not\in\mathcal{T}^*$, then $\mathcal{T}$ is not realizable; this assertion is immediate from 2.3. □

Suppose $\mathcal{G}$ is a family of fields and $\mathcal{G}^*$ is a set of isomorphism-class representatives of the elements of $\mathcal{G}$. In 3.8 we asked whether the families $\mathcal{G}$ and $\mathcal{G}^*$ are simultaneously realizable. In the case of a family of prime fields, Theorem 4.15 implies that this question has an affirmative answer. We state this result formally as Corollary 4.16.

Corollary 4.16. *If $\mathcal{T}$ is a family of prime fields and if $\mathcal{T}^*$ is a set of isomorphism-class representatives of the elements of $\mathcal{T}$, then $\mathcal{T}$ and $\mathcal{T}^*$ are simultaneously realizable.* □

4.17. Example 2.9 presents nonisomorphic zero-dimensional reduced rings R and S with $\mathcal{F}(R)=\mathcal{F}(S)$. The ring S contains two maximal ideals M and P that are not finitely generated, and $S/M\simeq S/P\simeq\mathrm{GF}(2)$. We remark that if S_0 is the subring of S obtained by gluing M and P over $M\cap P$, then R and S_0 are isomorphic. One could ask whether this example generalizes, in the sense that if $\mathcal{T}$ and $\mathcal{U}$ are zero-dimensional reduced rings with $\mathcal{F}(T)=\mathcal{F}(U)$, there exist isomorphic subrings T_0 and U_0 obtained from T and U by appropriate finite gluings. This question has a negative answer, as can be seen by the following example. Let p and q be prime, let $\mathcal{T}=\{\mathrm{GF}(p)\}\cup\{\mathrm{GF}(p^{q^i})\colon i>1\}$, and let

$\mathcal{U} = \{\mathrm{GF}(p^{q^i}): i \geq 1\}$. Let T^* and U^* be zero-dimensional reduced rings obtained as in Theorem 2.7 so that $\mathcal{F}(T^*) = \mathcal{T}$ and $\mathcal{F}(U^*) = \mathcal{U}$. If $T = T^* \oplus \mathrm{GF}(p^q)$ and $U = U^* \oplus \mathrm{GF}(p)$, then $\mathcal{F}(T) = \mathcal{F}(U) = \{\mathrm{GF}(p^{q^i}): i \geq 0\}$, so no gluing of maximal ideals is possible on either T or U. However, T and U are not isomorphic because, for example, the unique maximal ideal of U with residue field $\mathrm{GF}(p)$ is finitely generated, but the maximal ideal of T with the same residue field is not finitely generated.

5. Realizable families $\{K_\alpha\}$ with K_α absolutely algebraic of nonzero characteristic

We have already noted that in investigating realizability of a family $\{K_\alpha\}$, an important special case is that in which all of the K_α have the same characteristic c. In this section we consider the case where $c > 0$ and each K_α is absolutely algebraic (that is, is algebraic over its prime subfield). This is the case that sparked our initial interest in the topic of realizable families of fields, for if R is a zero-dimensional ring with $\mathcal{F}(R) = \{K_\alpha\}$ as described, then the prime subring π of R is isomorphic to $\mathrm{GF}(c)$, R is integral over π, and hence R is hereditarily zero-dimensional [8]. In the case at hand we may assume that each K_α is a subfield of an algebraic closure of $\mathrm{GF}(c)$; if R is diagonally imbedded in $T = \prod_\alpha K_\alpha$, then R is contained in the integral closure of $\mathrm{GF}(c)$. Corollary 5.2 provides a description of the integral closure of $\mathrm{GF}(c)$ in T.

Proposition 5.1. *Suppose $\{K_\alpha\}_{\alpha \in A}$ is a family of subfields of a field L. Let K be a subfield of $\bigcap K_\alpha$, and also denote by K the diagonal imbedding of K in $T = \prod_\alpha K_\alpha$. An element $b = \{b_\alpha\}$ of T is integral over K if and only if each b_α is algebraic over K and b has only finitely many distinct coordinates.*

Proof. If b is integral over K and if $f(X)$ is a monic polynomial over K having b as a root, then $0 = f(b) = \{f(b_\alpha)\}$, so $f(b_\alpha) = 0$ for each α. Hence each b_α is algebraic over K, and since $f(X)$ has only finitely many roots in L, b has only finitely many distinct coordinates.

If, conversely, each b_α is algebraic over K and if $\{b_{\alpha_1}, b_{\alpha_2}, \ldots, b_{\alpha_s}\}$ is the set of distinct coordinates of b, then b satisfies the monic polynomial $f = f_1 f_2 \cdots f_s \in K[X]$, where f_i is the minimal polynomial for b_{α_i} over K. In particular, b is integral over K. □

In Proposition 5.1, suppose $K_\alpha = L$ for each α and denote by L the diagonal imbedding of L in T. In this case Proposition 5.1 shows that the integral closure of K in T can be described as $L[\{e_\gamma: e_\gamma$ is an idempotent element of $T\}]$; this ring consists of all finite linear combinations $c_1 e_{\gamma_1} + \cdots + c_s e_{\gamma_s}$, where $c_i \in L$ for each i.

Corollary 5.2. *Let L be an algebraic closure of* GF(p), *let β be an infinite cardinal, and let T be a product of β copies of L. Thus we write $T = \prod_{\alpha \in A} L_\alpha$, where $L_\alpha = L$ for each α and $|A| = \beta$. Let $\{K_i\}_{i=1}^{\infty}$ be an ascending sequence of finite subfields of L such that $L = \bigcup_{i=1}^{\infty} K_i$ and let π be the prime subring of T.*

(1) *The integral closure π' of π in T is $\bigcup_{i=1}^{\infty} T_i$, where $T_i = K_i^{\beta}$ is the set of elements $\{b_\alpha\}$ in T with $b_\alpha \in K_i$ for each α.*

(2) *For each $\alpha \in A$, let F_α be a subfield of L_α and let $R = \prod_{\alpha \in A} F_\alpha$. The integral closure of π in R is*

$$R \cap \pi' = \bigcup_{i=1}^{\infty} (R \cap T_i) = \bigcup_{i=1}^{\infty} \Big[\prod_{\alpha \in A} (F_\alpha \cap K_i) \Big] .$$

Proof. (1) An element of T_i has at most $|K_i|$ distinct coordinates, and hence is integral over π by Proposition 5.1. Conversely, if $x = \{x_\alpha\} \in \pi'$, then x has only finitely many distinct coordinates, so some K_i contains each x_α. Hence $x \in T_i$.

It is clear that (2) follows from (1). □

The next result concerns residue fields of a product of fields, such as the products $\prod_{\alpha \in A}(F_\alpha \cap K_i)$ that arise in (2) of Corollary 5.2.

Theorem 5.3. *Let the notation be as follows: $\mathcal{F} = \{K_\alpha\}_{\alpha \in A}$ is a family of fields, $\mathcal{F}^*$ is a family of isomorphism-class representatives of elements of $\mathcal{F}$, $T = \prod_{\alpha \in A} K_\alpha$, and I is the direct sum ideal of T.*

(1) *If $\mathcal{F}^*$ contains only one element K and if K is finite, then each residue field of T is isomorphic to K.*

(2) *Each residue field of T is finite if and only if there exists a positive integer N such that $|K_\alpha| \le N$ for each α. If this condition is satisfied, then $\mathcal{F}^*$ is also a set of isomorphism-class representatives of the elements of $\mathcal{F}(T)$.*

(3) *Each free maximal ideal of T has infinite residue field if and only if, for each positive integer N, the set $\{\alpha \in A: |K_\alpha| \le N\}$ is finite.*

Proof. (1) Suppose $|K| = m$. Since K is imbedded in T, $|T/M| \ge m$ for each maximal ideal M of T. However, each element of T, and hence of T/M, satisfies the polynomial $X^m - X$ so that $|T/M| \le m$ as well. Hence $T/M \simeq K \simeq \mathrm{GF}(m)$, as asserted.

(2) If $\{|K_\alpha|\}$ is bounded above by the positive integer N, then $\mathcal{F}^* = \{F_i\}_{i=1}^{v}$ is a finite set of finite fields, and $\mathcal{F}$ can be partitioned into classes $\mathcal{F}_1, \ldots, \mathcal{F}_v$ under the equivalence relation of isomorphism, where each element of $\mathcal{F}_i$ is isomorphic to F_i. If $A_i = \{\alpha \in A: K_\alpha \in \mathcal{F}_i\}$ and $T_i = \prod\{K_\alpha: \alpha \in A_i\}$, then $T \simeq T_1 \oplus \cdots \oplus T_n$ and each residue field of T is isomorphic to a residue field of some T_i. Part (1) shows that, up to isomorphism, F_i is the only residue class of T_i. Therefore each residue field of T is finite and $\{F_i\}_{i=1}^{v} = \mathcal{F}^*$ is a set of isomorphism-class representatives of the elements of $\mathcal{F}(T)$.

Conversely, suppose that $\{|K_\alpha|\}$ is not bounded. If some K_α is infinite, it is clear that T has an infinite residue field. If each K_α is finite, we can choose a sequence $\{\alpha_i\}_{i=1}^\infty$ in A so that $|K_{\alpha_i}| > i$ for each i. Then since $\prod_\alpha K_{\alpha_i}$ is isomorphic to a direct summand of T, it suffices to consider the case where $A = \mathbb{Z}^+$ and $|K_i| > i$ for each i. In this case we show that each free maximal ideal of T has infinite residue field. (This is a special case of (3).) Suppose, to the contrary, that there exists a free maximal ideal M of T of finite index, say $|T/M| = u$. For $i \geq u$, let $\{a_{ij}\}_{j=1}^{u+1}$ be a subset of K_i with $u+1$ distinct elements and for $1 \leq j \leq u+1$, let $h_j = \sum_{i=u}^\infty a_{ij}e_i$. For some $j \neq k$, $h_j - h_k = \sum_{i=u}^\infty (a_{ij} - a_{ik})e_i \in M$. Because M is free, M contains the element $e_1 + \cdots + e_{u-1}$ of I, and hence $y = e_1 + \cdots + e_{u-1} + (h_j - h_k) \in M$. But y is a unit of T, and this is a contradiction.

(3) A slight modification of the argument given in the preceding paragraph shows that if the set $\{\alpha \in A: |K_\alpha| \leq N\}$ is finite for each $N \in \mathbb{Z}^+$, then each free maximal ideal of T has infinite residue field. For the converse, suppose that $U = \{\alpha \in A: |K_\alpha| \leq N\}$ is infinite for some N. If $S = \prod_{\alpha \in U} K_\alpha$, then (2) shows that each residue field of S is finite. Now S is a direct summand of T, say $T = S \oplus J$, where $J = \prod_{\alpha \in A-U} K_\alpha$. If M is a free maximal ideal of S, then $M + J$ is a free maximal of T of finite index, and this completes the proof. □

Theorem 5.4. *Let the notation L, $\{K_i\}_{i=1}^\infty$, $\{F_\alpha\}_{\alpha\in A}$, R and π be as in the statement of Corollary 5.2. Let $d \in \mathbb{Z}^+$, let $E = \mathrm{GF}(p^d)$, and let π^* be the integral closure of π in R. Then $E \in \mathcal{F}(\pi^*)$ if and only if for all sufficiently large i, the set $B_i = \{\alpha \in A: F_\alpha \cap K_i = E\}$ is nonempty.*

Proof. Suppose first that E is a residue field of π^*, say $E = \pi^*/M$, where M is maximal in π^*. For each $i \in \mathbb{Z}^+$, let R_i be the subring $\prod_{\alpha\in A}(F_\alpha \cap K_i)$ of π^*. Since (1) $R_i \subseteq R_{i+1}$ for each i, (2) $\pi^* = \bigcup_{i=1}^\infty R_i$, and (3) E is finite, there exists $n \in \mathbb{Z}^+$ such that E is a residue field of R_i for each $i \geq n$. However, since $\{|F_\alpha \cap K_i|: \alpha \in A\}$ is bounded above by $|K_i|$, part (2) of Theorem 5.3 shows that the residue fields of R_i are, up to isomorphism, precisely the fields $F_\alpha \cap K_i$ for $\alpha \in A$. Hence $E \in B_i$ for each $i \geq n$.

Conversely, suppose $n \in \mathbb{Z}^+$ is such that $B_i \neq \emptyset$ for each $i \geq n$. We observe that if $i \geq k \geq n$, then $B_i \subseteq B_k$. To see this note that if $\alpha \in B_i$, then $E = F_\alpha \cap L_i \supseteq F_\alpha \cap L_k \supseteq F_\alpha \cap L_n$. Moreover, since $E \in B_n$ we have $E = F_{\alpha_0} \cap L_n$ for some $\alpha_0 \in A$ so that $E \subseteq L_n$ and $E \subseteq F_\alpha \cap L_n$. Consequently, $E = F_\alpha \cap L_k$ and $\alpha \in B_k$. For $i \geq n$, we let J_i be the ideal of R_i consisting of the product of those fields $F_\alpha \cap K_i$ such that $\alpha \not\subseteq B_i$—that is, J_i consists of all tuples $\{b_\alpha\} \in R_i = \prod_{\alpha\in A}(F_\alpha \cap K_i)$ such that $b_\alpha = 0$ for each $\alpha \in B_i$. Because $B_i \supseteq B_{i+1}$ for $i \geq n$, $J_i \subseteq J_{i+1}$ for $i \geq n$. Moreover, since $B_i \neq \emptyset$ for $i \geq n$, J_i is a proper ideal of R_i for $i \geq n$, so $J = \bigcup_{i=n}^\infty J_i$ is a proper ideal of π^*. For $i \geq n$, R_i/J_i is a product of copies of E, so by part (1) of Theorem 5.3, each residue field of R_i/J_i is isomorphic to E. Thus, if P is a maximal ideal of π^* containing J, then $\pi^*/P \simeq E$, so E is a residue field of π^*. □

Corollary 5.5. *Suppose L is an algebraic closure of* $\mathrm{GF}(p)$. *Let $\{K_i\}_{i=1}^{\infty}$ be an ascending sequence of finite subfields of L such that $L = \bigcup_{i=1}^{\infty} K_i$ (for example, we could take $K_i = \mathrm{GF}(p^{i!})$), and let $\{F_\alpha\}_{\alpha\in A}$ be a realizable family of subfields of L. If E is a finite subfield of L such that no subfield of E belongs to $\{F_\alpha\}_{\alpha\in A}$ and if $N \in \mathbb{Z}^+$ is such that $E \subseteq K_N$, then for each $i \geq N$, the set $B_i = \{\alpha \in A: F_\alpha \cap K_i = E\}$ is empty.*

Proof. Let S be a zero-dimensional reduced ring such that $\mathcal{F}(S) = \{F_\alpha: \alpha \in A\}$, and consider S as diagonally imbedded in $R = \prod_{\alpha\in A} F_\alpha$. The prime subring π of R is isomorphic to $\mathrm{GF}(p)$, and since each residue field of S is integral over π, S is integral over π. That is, $S \subseteq \pi^*$, the integral closure of π in R. If E were a residue field of π^*, a subfield of E would be a residue field of S. Since, by assumption, no subfield of E is in $\{F_\alpha\} = \mathcal{F}(S)$, we conclude that $E \not\in \mathcal{F}(\pi^*)$. The proof of Theorem 5.4 then shows that B_i is empty for each $i \geq N$. □

Corollary 5.6. *Let $\{F_\alpha: \alpha \in A\}$ be a family of absolutely algebraic fields of characteristic $p \neq 0$. If $\{F_\alpha\}$ is realizable and if* $\mathrm{GF}(p) \not\in \{F_\alpha\}$, *then there exists a finite set $\{q_1, q_2, \ldots, q_v\}$ of primes such that each F_α contains one of the fields* $\mathrm{GF}(p^{q_1}), \ldots, \mathrm{GF}(p^{q_v})$.

Proof. For $i \in \mathbb{Z}^+$, let $K_i = \mathrm{GF}(p^{i!})$. Then $K_i \subseteq K_{i+1}$ for each i and $\bigcup_{i=1}^{\infty} K_i = L$. By Corollary 5.5, there exists $N \in \mathbb{Z}^+$ such that the set $\{\alpha \in A: F_\alpha \cap L_N = \mathrm{GF}(p)\}$ is empty. Let $q_1, q_2, \ldots, q_v$ be the prime divisors of $N!$. Because $\{\mathrm{GF}(p^{q_1}), \ldots, \mathrm{GF}(p^{q_v})\}$ is the set of minimal proper extensions of $\mathrm{GF}(p)$ in $\mathrm{GF}(p^{N!})$, each F_α contains one of these fields. □

6. Uniqueness of realizability of a family of fields

Example 2.9 shows that there exist nonisomorphic zero-dimensional reduced rings R and S such that $\mathcal{F}(R) = \mathcal{F}(S)$. Indeed, the rings R and S of this example are such that $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ are not homeomorphic. In this connection it is natural to consider a topological structure on a realizable family in the following sense. If $\mathcal{F} = \{K_\alpha\}_{\alpha\in A}$ is a realizable family of fields, say $\mathcal{F} = \mathcal{F}(R)$, then there is an associated bijection $K_\alpha \to M_\alpha$ between $\mathcal{F}$ and $\mathrm{Spec}(R)$ such that $K_\alpha \cong R/M_\alpha$ for each α. The Zariski topology of $\mathrm{Spec}(R)$ induces a topology on $\mathcal{F}$ via this bijection, and when we speak of a topology on $\mathcal{F}$, we understand a topology induced by such a bijection.

6.1. We remark that in general if $\mathcal{F} = \mathcal{F}(R)$, where R is zero-dimensional, then it is well known that $\mathrm{Spec}(R)$ (and therefore $\mathcal{F}$) is Hausdorff, compact, and totally disconnected (cf. [1, Exercise 11, p. 44]).

Two questions that arise naturally concerning a realizable family of fields $\mathcal{F}$ equipped with a topology via a bijection with Spec(R) are the following:

Question 6.2. For what realizable families $\mathcal{F}$ of fields is the topology defined on $\mathcal{F}$ by means of a realization $\mathcal{F} = \mathcal{F}(R)$ independent of R?

Question 6.3. If R and S are zero-dimensional reduced rings such that $\mathcal{F}(R) = \mathcal{F} = \mathcal{F}(S)$ by means of an identification that defines the same topology on $\mathcal{F}$, under what conditions does it follow that R and S are isomorphic as rings?

Concerning Question 6.2, if the family $\mathcal{F}$ is finite, then $\mathcal{F}$ necessarily has the discrete topology. If $\mathcal{F}$ is an infinite realizable family of fields with only one element of characteristic zero and only finitely many of characteristic p for each prime p, then there exists a unique topology for $\mathcal{F}$ so that $\mathcal{F} \cong \mathrm{Spec}(R)$, where R is zero-dimensional and reduced. To see this assertion, we observe that it follows from 4.5 that every prime ideal of R for which the residue field is of characteristic $p > 0$ is finitely generated. Since Spec(R) is infinite, there must exist at least one prime of R that is not finitely generated, so we conclude that the unique prime ideal Q of R such that R/Q is of characteristic zero is not finitely generated. By Proposition 4.6, the open subsets of Spec(R) containing Q are the cofinite subsets of Spec(R) containing Q. Thus Spec(R) as a topological space is the one-point compactification of the infinite discrete space $\mathrm{Spec}(R) - \{Q\}$.

There are other infinite realizable families of fields $\mathcal{F}$ for which the topology defined on $\mathcal{F}$ by means of a realization $\mathcal{F} = \mathcal{F}(R)$ is independent of R. For example, it can be shown that if $\{p_i\}$ is an infinite set of distinct prime integers and $\mathcal{F}$ consists of one copy of the Galois field with 2 elements and one copy of the field with 2^{p_i} elements for each i, then $\mathcal{F}$ is realizable with a unique topology. Another family with a unique topology is $\mathcal{F} = \{K_i\}_{i=0}^{\infty}$, where $K_0 = \mathbb{Q}$ and where the K_i, $i > 0$, are distinct quadratic extensions of $\mathbb{Q}$. Thus if $\{p_i\}$ is an infinite set of prime integers, then the family $\mathcal{F} = \{\mathbb{Q}(\sqrt{p_i})\}_{i=1}^{\infty} \cup \{\mathbb{Q}\}$ has a unique topology. Our method for proving these assertions is to first observe that for these families $\mathcal{F}$, if $\mathcal{F} = \mathcal{F}(R)$, then R is an integral extension of a subfield. Therefore R can be expressed as a directed union of Artinian subrings. The hypothesis on $\mathcal{F}$ then implies that there is a unique prime ideal of R that is not finitely generated, this being the prime of R having the unique minimal element of $\mathcal{F}$ as its residue field.

Question 6.3 is considered by Popescu and Vraciu in [12]. Example 5.3 of [12] asserts the existence of nonisomorphic zero-dimensional reduced rings R and S such that $\mathcal{F}(R) = \mathcal{F} = \mathcal{F}(S)$ by means of an identification that defines the same topology on $\mathcal{F}$, but this assertion with regard to Example 5.3 of [12] seems to us to be incorrect. The example may be described as follows: let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers topologized so that each positive integer is open in $\mathbb{N}$ and such that the open sets about 0 are the complements in $\mathbb{N}$ of finite sets of positive integers. Consider the family of fields $\mathcal{F} = \{k_n : n \in \mathbb{N}\}$, where $k_0 = \mathbb{Q}(i) = \mathbb{Q}[X]/(X^2+1)$, and where k_n, $n > 0$, is described as follows: let $\{F_n\}_{n=1}^{\infty}$ be an enumeration of the finite prime fields. If the polynomial X^2+1 is irreducible in $F_n[X]$, let $k_n = F_n[X]/(X^2+1)$; otherwise let $k_n = F_n$. Let a_n, b_n denote the roots of X^2+1 in k_n. Let A be a zero-dimensional reduced ring such that $\mathcal{F}(A) = \{F_n : n \in \mathbb{N}\}$, where $F_0 = \mathbb{Q}$. Regard A as a subring of $\prod_{n=0}^{\infty} F_n \subset \prod_{n=0}^{\infty} k_n = T$, and take $a = (a_n)$ and $b = (b_n)$ in T. The assertion in [12] is that the rings $A[a]$ and $A[b]$ are not isomorphic. But in fact, as subrings of T, $b = -a$ and $A[a] = A[b]$.

Roger Wiegand has obtained an example that does show the existence of two nonisomorphic zero-dimensional reduced rings R and S such that $\mathcal{F}(R) = \mathcal{F} = \mathcal{F}(S)$ by means of an identification that defines the same topology on $\mathcal{F}$. Therefore the conclusion drawn in [12] on the basis of Example 5.3 of [12] is correct. We are grateful to Roger for allowing us to include his example below, and for other helpful comments in regard to this paper.

Example 6.4 (Roger Wiegand). Let $K = \mathbb{Q}(\sqrt{2})$ and let L be an extension field of K having the property that any automorphism of L restricts to the identity map on K. For example, one choice of L is the field $\mathbb{R}$ of real numbers. For each positive integer n, define $\sigma_n : K \to L$ to be the $\mathbb{Q}$-homomorphism such that $\sigma_n(\sqrt{2}) = (-1)^n\sqrt{2}$. Let $T = \prod_{i=1}^{\infty}(L_i)$, where $L_i \cong L$ for each i. Let

$$R = \{(a_1, a_2, \ldots) \in T : \text{there exists } b \in K \text{ with } a_n = b \text{ for } n \gg 0\}\ ,$$

and let

$$S = \{(a_1, a_2, \ldots) \in T : \text{there exists } b \in K \text{ with } a_n = \sigma_n(b) \text{ for } n \gg 0\}\ .$$

Then both Spec(R) and Spec(S) are homeomorphic to the one-point compactification of $\mathbb{N} = \{1, 2, \ldots,\}$ with the discrete topology, and both have residue fields $L, L, \ldots, K$.

Suppose $f : R \to S$ is an isomorphism. Then f takes each maximal ideal of R with residue field L to such a maximal ideal of S. Let $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$ be these maximal ideals of R and S, respectively. (Thus $P_n = \{(a_1, a_2, \ldots) \in R : a_n = 0\}$.) Suppose $f(P_n) = Q_{\pi(n)}$, where π is a permutation of $\mathbb{N} = \{1, 2, \ldots\}$. There is an induced automorphism $f_n : R/P_n \to S/Q_{\pi(n)}$. Let $t = (\sqrt{2}, \sqrt{2}, \ldots)$ be the constant sequence $\sqrt{2}$ in R, and let $f(t) = (a_1, a_2, \ldots)$ in S. Since $f(t)^2 = 2$, we know that $a_i = \pm\sqrt{2}$. Furthermore, for $n \gg 0$ we have $a_{n+1} = -a_n$. Choose $n \gg 0$ such that $a_n = -\sqrt{2}$, and say $n = \pi(m)$. Then $f_m : R/P_m \to S/Q_n$ takes $\sqrt{2}$ to $-\sqrt{2}$. This contradicts the fact that any automorphism of L restricts to the identity map on K. We conclude that R and S are not isomorphic.

Note added in proof

In conversations with Roger Wiegand we have discovered that the proof given for Theorem 4.11 is incomplete. The status of Theorem 4.11 is open, and hence the assertions in Corollary 4.12 and Theorem 4.14 about realizability involving a pure transcendental extension of the field of rational numbers are also open. Roger and Sylvia Wiegand have indicated in conversation that these results appear to be correct, but the proof of Theorem 4.11 needs to be modified.

References

[1] M.F. Atiyah and I.G. Macdonald, Introduction to Commutative Algebra (Addison-Wesley, Reading, MA, 1969).
[2] W.D. Burgess and P. Menal, On strongly π-regular rings and homomorphisms into them, Comm. Algebra 16 (1988) 1701–1725.
[3] A.M.S. Doering and Y. Lequain, The gluing of maximal ideals—spectrum of a Noetherian ring—going up and going down in polynomial rings, Trans. Amer. Math. Soc. 260 (1980) 583–593.
[4] L. Fuchs and I. Halperin, On the imbedding of a regular ring in a a regular ring with identity, Fund. Math. 54 (1964) 285–290.
[5] L. Gillman and M. Jerison, Rings of Continuous Functions (Van Nostrand, New York, 1960).
[6] R. Gilmer and W. Heinzer, Some countability conditions on commutative ring extensions, Trans. Amer. Math. Soc. 264 (1981) 217–234.
[7] R. Gilmer and W. Heinzer, Jónsson ω_0-generated algebraic field extensions, Pacific J. Math. 128 (1987) 81–116.
[8] R. Gilmer and W. Heinzer, Products of commutative rings and zero-dimensionality, Trans. Amer. Math. Soc. 331 (1992) 663–680.
[9] R. Gilmer and W. Heinzer, Zero-dimensionality in commutative rings, Proc. Amer. Math. Soc., to appear.
[10] H.B. Mann, Introduction to Algebraic Number Theory (The Ohio State University Press, Columbus, OH, 1955).
[11] R.S. Pierce, Minimal regular rings, in: Warfield Memorial Volume, Contemporary Mathematics (American Mathematical Society, Providence, RI), to appear.
[12] N. Popescu and C. Vraciu, Sur la structure des anneaux absoluments plat commutatifs, J. Algebra 40 (1976) 364–383.
[13] L.C. Washington, Introduction to Cyclotomic Fields (Springer, Berlin, 1982).
[14] R. Wiegand, Globalization theorems for locally finitely generated modules, Pacific J. Math. 39 (1971) 269–274.
[15] R. Wiegand, Modules over universal regular rings, Pacific J. Math. 39 (1971) 807–819.